Associative memory neural networks with asymmetric connections

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Abstract—Static and dynamical properties of associative memory analog networks with asymmetric couplings for sequence processing are studied using the overlap dynamics with a finite number of patterns and the SCSNA for the extensive loading case. Several types of correlated attractors are found to occur and summarized into the phase diagram.

I. INTRODUCTION

There is growing interest in the study of attractor neural networks exhibiting a rich variety of dynamical behavior. Associative memory models without Lyapunov functions, which are not only of interest from the view point of uncovering new features in statistical-mechanics-related memory retrieval properties but also of importance in understanding physiological mechanisms of highly integrated information processing involving memorization in living nervous systems, are however, far less studied than networks with the energy concept[6, 5, 17, 14, 15].

Recently we have studied associative memory models that have several types of asymmetric synaptic couplings added to the auto-associative type (Hebb type) couplings: a) synaptic couplings memorizing (i) presynaptic activity and (ii) post-synaptic activity[8, 9, 11], b) asymmetrically diluted couplings[10], c) synaptic couplings for processing pattern sequences. For analyzing statistical behaviors of memory retrieval of those networks with extensively many stored patterns, we have made good use of the method of the self-consistent signal-to-noise analysis (SCSNA)[12, 13], which self-consistently decomposes the local field of a neuron into signal, noise, and output-proportional terms to obtain the set of order-parameter equations and is widely applicable[16] even for networks without the energy concept.

Main focus of the present article is on the retrieval properties of networks with asymmetric couplings for sequence processing together with its relation to a new kind of attractors called the correlated attractors[1] which have recently been proposed to be distinguished from the well-known retrieval and mixture states. In what follows we assume the neural network dynamics to be governed by the time-continuous analog network equations for the outputs of N neurons with a transfer function $F(\cdot)[13]$:

$$\frac{d\mathbf{z}_i}{dt} = -\mathbf{z}_i + F \left( \sum_j J_{ij} \mathbf{z}_j \right) \quad i = 1, \ldots, N. \quad (1)$$

II. ASYMMETRIC COUPLINGS FOR SEQUENCE PROCESSING AND CORRELATED ATTRACTION

An associative memory neural network with correlated attractors was proposed by Griniasy et al.[1] in an attempt to try to explain the experimental findings of Miyashita and Chang[2], which suggested, in addition to a manifestation of attractor dynamics in the network of a monkey's neurons, a conversion of temporal order between stimuli to spatial correlations between attractors.

Theoretical studies[1, 3, 4] conducted so far have been based on a modified Hopfield type network that has symmetric couplings between neurons for the sake of simplicity of the analysis. They have indicated that correlated attractors, which, generated corresponding to uncorrelated patterns of stimuli, still have correlations between attractors associated to temporally close stimuli in the learning stage, can be described by a characteristic overlap vector $\mathbf{m} = (m_1, \ldots, m_p)$ reflecting their spatial correlations with a short correlation distance in the stored sequence: the overlap vector of $0$-temperature model with a finite number of stored patterns $p$ takes the form $\mathbf{m} = \frac{1}{\sqrt{p}} (77, 51, 13, 3, 1, 0, \ldots, 0, 1, 3, 13, 51)$ ($\equiv \mathbf{m}_{CA}$), which is universal for $p \geq 11$ in the sense that the nonzero components do not depend either on $p$ or the parameter of the symmetric couplings employed. The values for the overlap vector remains almost unchanged even in the case of extensive loading. It is of interest to explore how the correlated attractors behave when non-symmetric synaptic couplings are assumed for attractor networks of associative memory.

In our model for networks with asymmetric synaptic couplings exhibiting correlated attractors, we assume that $S$ learned random patterns $\xi_{il}^\mu$ ($\xi_{il}^\mu = \pm 1, i = 1, \ldots, N, \mu = 1, \ldots, S$) for each learning stage $l$ ($l = 1, \ldots, L$) represent a cyclic sequence. Then $J_{ij}$ is constructed so as to couple $S$ consecutive patterns for each $l$ with a strength $1 - a$ in addition to
the auto-associative Hebb type term with a strength $a$ ($0 \leq a \leq 1$):

$$J_{ij} = \frac{a}{N} \sum_{l=1}^{L} \sum_{\mu=1}^{S} \xi_{ij}^{\mu} \xi_{ij}^{\mu}$$

$$+ \frac{1-a}{N} \sum_{l=1}^{L} \sum_{\mu=1}^{S} \left\{ b \xi_{ij}^{\mu+1} \xi_{ij}^{\mu} + (1-b) \xi_{ij}^{\mu-1} \xi_{ij}^{\mu} \right\},$$

$$J_{ii} = 0,$$

(2)

where $b$ and $1-b$ ($0 \leq b \leq 1$) respectively denote a relative strength of asymmetric forward and backward projections responsible for sequential memory recall, and a periodic condition $\xi_{ij}^{S+1} = \xi_{ij}^{1}$ is imposed. The number of total memory patterns is then given by $p = LS$ with the loading rate $\alpha = LS/N$.

A. Case with $\alpha = 0$

When $p$ is kept finite in the limit $N \to \infty$, the dynamics for the pattern-overlaps $m_{\mu} = \frac{1}{N} \sum_{i} \xi_{i}^{\mu} x_{j}$ can easily be obtained by using the recipe of the sub-lattice decomposition of the total $N$ neurons of the system. For simplicity, setting $L = 1$ one obtains[13]

$$\frac{dm_{\mu}}{dt} = -m_{\mu} + 2-\rho \sum_{\xi} \xi_{\mu} F \left( h \left( \xi_{\mu}, \xi \right) \right),$$

$$\mu = 1, \ldots, p,$$

(3)

$$h \left( \xi_{\mu}, \xi \right) = \sum_{\nu=1}^{p} \left[ a \xi_{\nu} + (1-a) \left\{ b \xi_{\nu+1} + (1-b) \xi_{\nu-1} \right\} \right],$$

which exhaustively describe the dynamical as well as static behavior of the present networks. The case with $b = 1$ was previously studied[7]. Fixed-point-type attractors are obtained by setting $\frac{dm_{\mu}}{dt} = 0$:

$$m_{\mu} = 2-\rho \sum_{\xi} \xi_{\mu} F \left( h \left( \xi_{\mu}, \xi \right) \right), \quad \mu = 1, \ldots, p.$$  

(5)

They are classified mainly into three types of attractors in the case where the transfer function is assumed to be $F(u) = \text{sgn}(u)$: retrieval state, mixture one, and the so-called correlated attractor. Assuming that pattern 1 is chosen as a particular pattern to be retrieved, the retrieval state is defined as $\xi_{\mu} = (1,0,\ldots,0)$ and mixture states are given typically by $(1,1,\ldots,1)$ and some variants of it. The correlated attractors are of the form described earlier.

We display in Fig. 1 the phase diagram with $S = 11$ on the $a - b$ plane showing each region for the above three types of fixed-point attractors together with the region exhibiting oscillatory motions. The retrieval states, which have large basin of attraction, occur for $a > 1/2$. At least three types of correlated attractors $CA, CA'$, and $CA''$ are found to appear depending on initial conditions for the dynamics (3). The $CA$ for which $\bar{\mu} = \bar{\mu}_{CA}$ is exactly the same as that found previously for the symmetric coupling case with $b = 1/2$ and is realized by starting the network with an initial state near pattern 1, that is, $\bar{\mu} = (\bar{m}_{1}(0),0,\ldots,0)$ with $\bar{m}_{1}(0) \approx 1$. The $CA'$ and $CA''$, which is respectively given by $\bar{\mu} = \frac{a}{b} (19,13,3,1,0,0,0,0,0,0,0)$ and $\bar{\mu} = \frac{a}{b} (19,13,3,1,0,0,0,0,0,0,0)$ have narrow basin of attraction and are only approached starting from quite near the attractor itself. In view of the experimental findings by Miyashita and Chang, the $CA$ and $CA''$ should be considered as irrelevant attractors, while the $CA$ is a relevant one. They coexist with the retrieval states in the $a > 1/2$ region. States with oscillatory motions are also found to coexist with the retrieval states, $CA'$, and $CA''$. Since the asymmetric couplings (2) are constructed so as to recall consecutive memory patterns, the pattern overlaps are seen to oscillate in order as shown in Fig. 2. When $S$ is chosen to be even, the oscillatory behavior appears in a wider region of the phase diagram occupying part of the region with $a > \frac{1}{2}$ where the oscillatory states are approached from a pure state corresponding to a particular pattern. The phase boundaries between the different regions for the attractors can be obtained analytically and the qualitative behavior of the network is almost similar to each other for $S \geq 11$ except for
the difference due to whether \( S \) is even or odd.

**B. Case with \( \alpha \neq 0 \)**

When an extensive number of patterns are stored with \( S \) fixed and \( L/N = \alpha/S \) in the limit \( N \to \infty \), static properties of memory retrieval of the network can be analyzed by the method of the SCSNA. We assume here, for the sake of simplicity, \( b = 1 \) in Eq. (2). Then the renormalized local field \( (S \geq 3) \) is obtained as

\[
h_i = \sum_{\mu} \left( \alpha \xi_i^{1+\mu} + (1 - \alpha) \xi_i^{1+\mu+1} \right) m_{1+\mu} + \xi_i + \Gamma x_i \quad \text{(6)}
\]

where

\[
\Gamma = \alpha \left\{ \Omega \left( a + (1 - a) R^{S-1} \right) - a \right\} , \quad \text{(7)}
\]

\[
\langle z^2 \rangle = \alpha \Omega^2 G(R) q \quad \text{(8)}
\]

with

\[
\Omega = \frac{1}{(1 - a U) \left\{ 1 - \left( \frac{(1-a)U}{1-aU} \right)^S \right\}} , \quad \text{(9)}
\]

\[
R = \frac{(1 - a) U}{1 - a U} , \quad \text{(10)}
\]

\[
G(R) = \frac{1}{1 - R^2} \left\{ \left( 1 - 2a + 2a^2 \right) \left( 1 - R^{2S} \right) \right. \\
+ \left( 1 - R^2 \right) 2a(1-a)R^{S-1} \\
+ \left( 1 - R^{2(S-1)} \right) 2a(1-a)R \right\} . \quad \text{(11)}
\]

The SCSNA eqs read

\[
m^{1+\mu} = \left\langle \xi_{i}^{1+\mu} Y(z) Dz \right\rangle (1+\mu) \quad \mu = 1, \ldots, S , \quad \text{(12)}
\]

\[
q = \left\langle Y^2 Dz \right\rangle , \quad \text{(13)}
\]

\[
\sqrt{\alpha \sigma U} = \left\langle z Y(z) Dz \right\rangle \quad \text{(14)}
\]

with \( Y(z) \) satisfying

\[
Y' = F \left( \sum_{\mu} (a \xi_{i}^{1+\mu} + (1 - a) \xi_{i}^{1+\mu+1}) m_{1+\mu} + \sqrt{\alpha \sigma U} + \nabla Y \right) \quad \text{(15)}
\]

and \( Dz \) being for the unit-gaussian integral.

We are interested in the problem of how the retrieval states behave in the presence of asymmetric forward projections under the use of an arbitrarily shaped transfer function including a nonmonotonic one. Setting \( S = 3 \) and \( F(u) \) to be of the end-cutoff-type

\[
F(u) = \begin{cases} 
0 & u < -\theta, \theta < u \\
-1 & -\theta < u < 0 \\
1 & 0 < u < \theta 
\end{cases} , \quad \text{(16)}
\]

we solve the SCSNA eqs. for \( m^\mu (\mu = 1, 2, 3, r, \text{ and } U) \). We depict in Fig. 3 an example of the behavior of the overlaps with change in \( \alpha \) in the case of the nonmonotonic transfer function with \( \theta = 0.7 \) obtained from the SCSNA together with numerical simulations. Note that although \( m^2 \) and \( m^3 \) remain to be nonzero, the states with \( m \) given in the Figure are the retrieval states corresponding to pattern 1.

Figure 4 represents the phase diagram showing the dependence of the storage capacities \( \alpha_c \) and \( \alpha_c \) on the parameter \( \theta \) when \( \alpha = 0.8 \) and \( b = 1 \). Compared to the network with the standard Hebb type couplings (i.e. \( a = 1 \)) not shown here), qualitatively the same behavior such as an enhancement of the storage capacity and the occurrence of instability of the SCSNA solution near the \( \alpha_c \) due to the nonmonotonicity of the transfer function are observed[13].

We have also obtained the SCSNA equations with arbitrary \( b \), for which the expression of noise variance \( \langle z^2 \rangle \) is too lengthy to write down. Based on those equations, behaviors of retrieval properties including the relevant and irrelevant types of correlated attractors that occur for \( a = 0 \) can be studied in terms of the phase diagrams for various types of transfer functions.

**III. CONCLUSIONS**

Considering time-continuous analog networks with asymmetric couplings of sequence processing type, we
have investigated the occurrence of various types of attractors including the correlated attractors for the retrieval dynamics of the pattern-overlaps with a finite number of stored patterns to present the phase diagrams on the space of parameters of the couplings. Relevant and irrelevant correlated attractors have been found to occur. In the case of extensive loading, we have employed the SCSSNA to analyze the behaviors of the retrieval solutions of the networks with the end-cutoff-type transfer function, the results of which have been in good agreement with the results of numerical simulations.

References


Figure 3: Plots of $m_1^1$, $m_2^1$, and $m_3^1$ as a function of $\alpha$ for the retrieval states with $S = 3$, $a = 0.8$, and $b = 1$ obtained from numerical simulations with $N = 1000$ together with the corresponding SCSSNA result for the end-cutoff-type transfer function with $\theta = 0.7$. The upper bound for the existence of the stable retrieval solutions defines the storage capacity $\alpha_c$. The deviation of $\alpha_c$ from the other upper bound $\tilde{\alpha}_c$ determined from the SCSSNA is due to the occurrence of instability.

Figure 4: Plots of $\tilde{\alpha}_c$ and $\alpha_c$ as a function of $\theta$ in the case of $S = 3$, $a = 0.8$, $b = 1$. 

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